COHERENT STATES QUANTIZATION OF GENERALIZED BERGMAN SPACES ON THE UNIT BALL OF $\mathbb{C}^n$ WITH A NEW FORMULA FOR THEIR ASSOCIATED BEREZIN TRANSFORMS

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ABSTRACT. While dealing with a class of generalized Bergman spaces on the unit ball, we construct for each of these spaces a set of coherent states to apply a coherent states quantization method. This provides us with another way to recover the Berezin transforms attached to these spaces. Finally, a new formula representing these transforms a functions of the Laplace-Beltrami operator is established in terms of Wilson polynomials by using the Fourier-Helgason transform.

1 INTRODUCTION

The Berezin transform introduced in [3] for certain classical bounded symmetric domains in $\mathbb{C}^n$ is a transform linking the Berezin symbols and symbols for Toeplitz operators. It is present in the study of the correspondence principle. The formula representing the Berezin transform as a function of the Laplace operators $\Delta_1, ..., \Delta_r$ (r being the rank of the domain) plays a key role in the Berezin quantization [4].

In this paper, we deal with the rank one symmetric domains. Namely the unit ball $\mathbb{B}^n$ in $(\mathbb{C}^n, \langle , \rangle)$ endowed with its Bergman metric. We are precisely concerned with the L^2-eigenspaces

$$\mathcal{A}_m^{2,\nu}(\mathbb{B}^n) = \left\{ \varphi \in L^2(\mathbb{B}^n, (1-|\xi|^2)^{-n-1} d\mu), H_\nu \varphi = \epsilon_m^{\nu,n} \varphi \right\} \tag{1.1}$$

associated to the discrete spectrum

$$\epsilon_m^{\nu,n} = 4\nu(2m+n) - 4m(m+n), m = 0, 1, 2, ..., [\nu - n/2] \tag{1.2}$$

of the Schrödinger operator with uniform magnetic field on $\mathbb{B}^n$ given by

$$H_\nu = -4(1-|z|^2)\left(\sum_{i,j=1}^{n} \left(\delta_{ij} - z_i \overline{z_j}\right) \frac{\partial^2}{\partial z_i \partial \overline{z}_j} + \nu \sum_{j=1}^{n} (z_j \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial \overline{z}_j}) + \nu^2 \right) + 4\nu^2 \tag{1.3}$$

provided that $\nu > n/2$. Above $[x]$ denotes the greatest integer not exceeding x. For $m \in \mathbb{Z}_+$, the Berezin transform associated with the space in (1.1) was obtained in [13] via the well known formalism of Toeplitz operators as

$$\begin{aligned} \mathfrak{B}_m^{\nu,n}[\varphi](z) &= \frac{m!\,(2\nu-2m-n)\,\Gamma(2\nu-m)\,\Gamma(n)}{n!\Gamma(2\nu-m-n+1)\,\Gamma(n+m)} \int_{\mathbb{B}} \left(\frac{(1-|z|^2\,(1-|\xi|^2)}{|1-\langle z,\xi\rangle|^2} \right)^{2(\nu-m)} \\ &\quad \times \left(P_m^{(n-1,2(\nu-m)-n)} \left(1-2|\xi|^2\right)\right)^2 \frac{\varphi(\xi)}{\left(1-|\xi|^2\right)^{n+1}} d\mu(\xi) \end{aligned} \tag{1.4}$$

where $P_m^{(\alpha,\beta)}(.)$ denotes the Jacobi polynomial [16]. Moreover this transform have been expressed as a function $f(\Delta_{\mathbb{B}^n})$ of the Laplace-Beltrami operator $\Delta_{\mathbb{B}^n}$ in terms of an ${}_3F_2$-sum, see (5.24) below. Our aim here is to construct for each of the eigenspaces in (1.1) a set of coherent states by following a generalized formalism [11] in order to apply a coherent states quantization method.

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This provides us with another way to recover the Berezin transforms in (1.4) attached to the L^2-eigenspace spaces in (1.1). Finally, we add a new formula expressing the transform (1.4) as a function of the Laplace-Beltrami operator. The idea is to make the integral (1.4) appear as "convolution product" of the function φ with a specific radial function given in terms of the square of a Jacobi polynomial. Next, a straightforward computation of the spherical transform of this radial function with the use of a Clebsh-Gordon type linearisation [8] for the square of a Jacobi polynomial amounts to a finite sum containing some integrals whose general form was given by Koornwinder [17] in terms of Wilson polynomials.

This paper is summarized as follows. In Section 2, we recall briefly the formalism of coherent states quantization we will be using. Section 3 deals with some needed facts on the generalized Bergman spaces. In Section 4, we construct for each of these spaces a set of coherent states and we apply the corresponding quantization scheme in order to recover their associated Berezin transforms. In Section 5, we present the formula expressing these Berezin transforms as functions of the Laplace-Beltrami operator by a different way and in a new form.

2 COHERENT STATES QUANTIZATION

Coherent states are mathematical tools which provide a close connection between classical and quantum formalism. In general, they are a specific overcomplete set of vectors in a Hilbert space satisfying a certain resolution of the identity condition. Here, we review a coherent states formalism starting from a measure space "*as a set of data*" as presented in [11]. Let $X = \{x \mid x \in X\}$ be a set equipped with a measure $d\mu$ and $L^2(X, d\mu)$ the space of $d\mu-$square integrable functions on X. Let $\mathcal{A}^2$ be a subspace of $L^2(X, d\mu)$ with an orthonormal basis $\left\{\Phi_j\right\}_{j=0}^{+\infty}$. Let $\mathcal{H}$ be another (functional) space with a given orthonormal basis $\left\{\phi_j\right\}_{j=0}^{+\infty}$. Then consider the family of states $\{\mid x >\}_{x\in X}$ in $\mathcal{H}$, through the following linear superposition:

$$\mid x >:= \left(\mathcal{N}\left(x\right)\right)^{-\frac{1}{2}} \sum_{j=0}^{+\infty} \Phi_j\left(x\right) \mid \phi_j >, \tag{2.1}$$

where

$$\mathcal{N}\left(x\right) = \sum_{j=0}^{+\infty} \Phi_j\left(x\right) \overline{\Phi_j\left(x\right)}. \tag{2.2}$$

These coherent states obey the normalization condition

$$\langle x \mid x \rangle_{\mathcal{H}} = 1 \tag{2.3}$$

and the following resolution of the identity of $\mathcal{H}$

$$\mathbf{1}_{\mathcal{H}} = \int_X \mid x >< x \mid N\left(x\right) d\mu\left(x\right) \tag{2.4}$$

which is expressed in terms of Dirac's bra-ket notation $\mid x >< x \mid$ meaning the rank-one -operator $\varphi \mapsto \langle \varphi \mid x \rangle_{\mathcal{H}} . \mid x >$. The choice of the Hilbert space $\mathcal{H}$ define in fact a quantization of the space X by the coherent states in (2.1), via the inclusion map $x \mapsto \mid x > \in \mathcal{H}$ and the property (2.4) is crucial in setting the bridge between the classical and the quantum mechanics. The Klauder-Berezin coherent states quantization consists in associating to a classical observable that is a function $f\left(x\right)$ on X having specific properties the operator-valued integral

$$A_f := \int_X \mid x >< x \mid f\left(x\right) \mathcal{N}\left(x\right) d\mu\left(x\right) \tag{2.5}$$

The function $f\left(x\right) \equiv \widehat{A}_f\left(x\right)$ is called upper (or contravariant) symbol of the operator A_f and is nonunique in general. On the other hand, the expectation value $\langle x \mid A_f \mid x \rangle$ of A_f with respect

to the set of coherent states $\{| x >\}_{x\in X}$ is called lower (or covariant) symbol of A_f. Finally, associating to the classical observable $f(x)$ the obtained mean value $\langle x \mid A_f \mid x\rangle$, we get the Berezin transform of this observable. That is,

$$B[f](x) := \langle x \mid A_f \mid x\rangle, \ x \in X. \tag{2.6}$$

For all aspect of the theory of coherent states and their genesis, we refer to the survey [9] by Dodonov or to the book by Gazeau [11].

3 The spaces $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$

In this section, we review some results on the L^2-concrete spectral analysis of the Schrödinger operator H_ν in (1.3) and acting in the Hilbert space $L^2(\mathbb{B}^n, d\mu_n)$,see [7], for more details. Let $\mathbb{B}^n = \{z \in \mathbb{C}^n; | z |< 1\}$ be the unit ball in $\mathbb{C}^n$ with the Lebesgue measure $d\mu$ normalized so that $\mu(\mathbb{B}^n)$ and let $\partial\mathbb{B}^n = \{\omega \in \mathbb{C}^n, | \omega |= 1\}$ be the unit sphere with $d\sigma$ the normalized measure on it. Let $G = SU(n, 1)$ be the group of all $\mathbb{C}$-linear transforms g on $\mathbb{C}^{n+1}$ that preserve the indefinite hermitian form $\sum_{j=1}^n | z_j |^2 - | z_{n+1} |^2$, with $\det g = 1$. Then G acts transitively on the unit ball by

$$G \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to g.z = (az + b)(cz + d)^{-1}. \tag{3.1}$$

As a homogeneous space we have the identification $\mathbb{B}^n = G/K$ where $K = S(U(n) \times U(1))$ is the stabilizer of 0. It is endowed with its usual Khaler-Bergman metric $ds^2 = -\sum_{i,j}^n \partial_j\overline{\partial}_j(Log(1 - |z|^2))dz_i \otimes \overline{dz_j}$. The Bergman distance and the volume element on $\mathbb{B}^n$ are given respectively by

$$\cosh^2 d(z, w) = \frac{|1 - \langle z, w\rangle|^2}{(1- | z |^2)(1- | w |^2)} \tag{3.2}$$

and $d\mu_n(z) = (1- | z |^2)^{-(n+1)} d\mu(z)$.
The group G acts unitarily on the space $L^2(\mathbb{B}^n, d\mu_n)$, via $U(g)F(z) = F(g^{-1}.z)$. Let consider the magnetic gauge vector potential given through the canonical 1-form on $\mathbb{B}^n$: $\theta = -i(\partial - \overline{\partial})Log(1- | z |^2)$, to which the Schrodinger operator

$$H_\nu = -(d + i\nu\mathrm{ext}(\theta))^* (d + i\mathrm{ext}(\theta)) + 4\nu^2 \tag{3.3}$$

can be associated. Here $\nu \geq 0$ is a fixed number d denotes the usual exterior derivative on differential forms on $\mathbb{B}^n$ and $\mathrm{ext}(\theta)$ is the exterior multiplication by θ while the symbol $*$ stands for the adjoint operation with respect to the Hermitian scalar product induced by the Bergman metric ds^2 on differential forms. Note that when $\nu = 0$, the operator in (3.3) reduces to

$$H_0 \equiv \Delta_{\mathbb{B}^n} = 4(1- | z |^2) \sum_{i,j=1}^n \left(\delta_{ij} - z_i\overline{z_j}\right) \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \tag{3.4}$$

which is the Laplace-Beltrami operator of the Bergman ball $\mathbb{B}^n$. For general $\nu \geq 0$, the Schrodinger operator H_ν in (3.3) can be expressed in the complex coordinates $(z_1, ..., z_n)$ by the formula (1.3) see [2],[6] and [12].

Now, for an arbitrary complex number λ, a fundamental family of eigenfunctions of H_ν with eigenvalue $\lambda^2 + 4\nu^2 + n^2$ is given by the Poisson kernels :

$$z \mapsto P_\lambda^\nu(z, \theta) = \left(\frac{1- | z |^2}{| 1- < z, \theta >|^2}\right)^{\frac{1}{2}(i\lambda+1)} \left(\frac{1 - \overline{< z, \theta >}}{1- < z, \theta >}\right)^\nu, z \in \mathbb{B}^n. \tag{3.5}$$

Moreover, a complete description of the expansion of an eigenfunction f of H_ν with eigenvalue $\lambda^2 + 4\nu^2 + n^2$, in terms of the appropriate Fourier series in $\mathbb{B}^n$ have been given in [7, Proposition

2.2]. Precisely,

$$f(z) = (1-\rho^2)^{\frac{i\lambda+n}{2}} \sum_{p,q=0}^{+\infty} \rho^{p+q} \cdot {}_2F_1\left(\frac{i\lambda+n}{2}+\nu+p, \frac{i\lambda+n}{2}-\nu+q, p+q+n; \rho^2\right) a_{p,q}^{\lambda,\nu}.h_{p,q}(\theta), \tag{3.6}$$

in $C^\infty([0,1[\times\partial\mathbb{B}^n)$, $z=\rho\theta$, $\rho\in[0,1[$ and $\mid\theta\mid=1$. Above ${}_2F_1$ denotes the Gauss hypergeometric function [14] and $a_{p,q}^{\lambda,\nu} = (a_{p,q,j}^{\lambda,\nu}) \in \mathbb{C}^{d(n,p,q)}$ are complex numbers, where

$$d(n,p,q) := \frac{(p+q+n-1)(p+n-2)!(q+n-2)!}{p!q!(n-1)!(n-2)!} \tag{3.7}$$

is the dimension of the space $H(p,q)$ of restrictions to the unit sphere $\partial\mathbb{B}^n$ of harmonic polynomials $h(z)$ on $\mathbb{C}^n$, which are homogeneous of degree p in z and degree q in $\bar{z}$, see [10] or [19] for more details. The notation "." in (3.6) means the following finite sum

$$a_{p,q}^{\lambda,\nu}.h_{p,q}(\theta) = \sum_{j=1}^{d(n,p,q)} a_{p,q,j}^{\lambda,\nu} h_{p,q}^{j}(\theta), \tag{3.8}$$

where $\{h_{p,q}^j\}_{1\le j\le d(n,p,q)}$ is an orthonormal basis of $H(p,q)$. The spectral analysis of H_ν have been studied by many authors, see [7] and references therein. Actually, H_ν is an elliptic densely defined operator on the Hilbert space $L^2(\mathbb{B}^n, (1-\langle z,z\rangle)^{-(n+1)}d\mu)$ admitting a unique self-adjoint realization also denoted by H_ν. Its spectrum consists of a continuous part given by $[n^2,+\infty[$ (corresponding to scattering states) and a finite number of infinitely degenerate eigenvalues $\epsilon_m^{\nu,n}$ given by (1.2) (characterizing bound states) provided that $2\nu>n$. More precisely, $\epsilon_m^{\nu,n} = \lambda_m^2 + 4\nu^2 + n^2$, with $\lambda_m = i(2m+n-2\nu)$, $m=0,1,...,[\nu-n/2]$. Here, we focus on the discrete part of the spectrum, which is labeled by the integer m and the corresponding eigenspace $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$ defined in (1.1). Taking into account (3.6) and expressing the involved hypergeometric in terms of Jacobi polynomial, an orthonormal basis of $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$ can be given explicitly by

$$\Phi_{p,q}^{\nu,m,j}(z) = \kappa_{p,q}^{\nu,m,n}\left(1-|z|^2\right)^{\nu-m} P_{m-q}^{(n+p+q-1,2(\nu-m)-n)}\left(1-2|z|^2\right) h_{p,q}^j(z,\bar{z}) \tag{3.9}$$

with

$$\kappa_{p,q}^{\nu,m,n} = \left(\frac{n\Gamma(2\nu-m-n-q+1)\Gamma(p+n+m)}{(m-q)!(2(\nu-m)-n)\Gamma(2\nu-m+p)}\right)^{-\frac{1}{2}}. \tag{3.10}$$

for varying $p=0,1,2,...$, $q=0,1,....,m$ and $j=1,....,d(n;p,q)$. Furthermore, the space $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$ is a reproducing kernel Hilbert space. That is, there exists a unique complex valued function $K^{\nu,m}$ on $\mathbb{B}^n\times\mathbb{B}^n$ such that, denoting $K_z^{\nu,m}(w) = K^{\nu,m}(w,z)$, $K_z^{\nu,m}$ belongs to $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$ for any $z\in\mathbb{B}^n$ and

$$f(z) = < f, K_z^{\nu,m} >,$$

for all functions f in $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$ and all $z\in\mathbb{B}^n$.Its expression can be given explicitly as function of the Bergman geodesic distance as

$$\begin{aligned} K^{\nu,m}(z,w) &= \frac{(2(\nu-m)-n)\Gamma(2\nu-m)}{n!\Gamma(2\nu-m-n+1)}\left(\frac{(1-\overline{<z,w>})}{1-<z,w>}\right)^{\nu} \\ &\quad\times(\cosh d(z,w))^{-2(\nu-m)}) P_m^{(n-1,2(\nu-m)-n)}(1-2\tanh^2 d(z,w)) \end{aligned} \tag{3.11}$$

Remark 3.1. For $m=0$, the space $\mathcal{A}_0^{2,\nu}(\mathbb{B}^n)$ reduces further to be isomorphic to the weighted Bergman space of holomorphic function ψ on $\mathbb{B}^n$ satisfying the growth condition

$$\int_{\mathbb{B}^n} |\psi(z)|^2 ((1-\langle z,z\rangle)^{2\nu-n-1} d\mu(z) < +\infty.$$

This fact justify why the eigenspace $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$ have been also called a *generalized Bergman spaces of index m*.

4 Coherent states quantization

Now, to adapt the defintion (2.1) of coherent states for the context of the generalized Bergman spaces in (1.1) we first list the following notations.

- $(X, d\eta) := \left(\mathbb{B}^n, \left(1-|z|^2\right)^{-(n+1)} d\mu\right), d\eta \equiv d\mu_n$ is the volume element on $\mathbb{B}^n$.
- $x \equiv z \in \mathbb{B}^n$.
- $\mathcal{A}^2 := \mathcal{A}_m^{2,\nu}(\mathbb{B}^n) \subset L^2(\mathbb{B}^n, \left(1-|z|^2\right)^{-n-1} d\mu)$.
- $\{\Phi_k(x)\} \equiv \left\{\Phi_{p,q,j}^{\nu,m}(z)\right\}$ is the orthonormal basis of $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$ in (3.8)
- $\mathcal{N}(x) \equiv \mathcal{N}(z)$ is a normalization factor.
- $\{\varphi_k\} \equiv \left\{\varphi_{p,q,j}\right\}$ is an orthonormal basis of another (functional) Hilbert space $\mathcal{H}$.

Definition 4.1. *For each fixed integer $m = 0, 1, ..., [n - \nu/2]$, a class of generalized coherent states associated with the space $A_m^{2,\nu}(\mathbb{B}^n)$ is defined according to (2.1) by the form*

$$\phi_z^{\nu,m} \equiv \mid z, \nu, m > := (\mathcal{N}(z))^{-\frac{1}{2}} \sum_{\substack{0\leq q\leq m, 0\leq p<+\infty \\ 1\leq j\leq d(n,p,q)}} \Phi_{p,q,j}^{\nu,m}(z)\, \varphi_{p,q,j} \tag{4.1}$$

where $\mathcal{N}(z)$ is a normalization factor.

Proposition 4.1. *The factor in (4.1) is given by*

$$\mathcal{N}(z) = \frac{(2(\nu - m) - n)\,\Gamma(2\nu - m)}{n!\Gamma(2\nu - m - n + 1)} \frac{\Gamma(m+n)}{m!\Gamma(n)} \tag{4.2}$$

for every $z \in \mathbb{B}^n$.

Proof. To calculate this factor, we start by writing the condition

$$\langle \phi_z^{\nu,m}, \phi_z^{\nu,m} \rangle_{\mathcal{H}} = 1. \tag{4.3}$$

Equation (4.3) is equivalent to

$$(\mathcal{N}(z))^{-1} \sum_{p=0}^{+\infty} \sum_{q=0}^{m} \sum_{j=1}^{d(n,p,q)} \Phi_{p,q,j}^{\nu,m}(z)\, \overline{\Phi_{p,q,j}^{\nu,m}(z)} = 1 \tag{4.4}$$

Making use of (3.9) and (3.11) for the particular case $z = w$, we get that

$$\mathcal{N}(z) = \frac{(2(\nu - m) - n)\,\Gamma(2\nu - m)}{n!\Gamma(2\nu - m - n + 1)} P_m^{(n-1, 2(\nu-m)-n)}(1) \tag{4.5}$$

Next, by the following fact on Jacobi polynomial [14]:

$$P_m^{(\alpha,\beta)}(1) = \frac{\Gamma(m + \alpha + 1)}{m!\Gamma(\alpha + 1)} \tag{4.6}$$

for $\alpha = n - 1$ to arrive at the announced result.The states $\phi_z^{\nu,m} \equiv \mid z, \nu, m >$ satisfy the resolution of the identity

$$1_{\mathcal{H}} = \int_{\mathbb{B}^n} \mid z, \nu, m >< z, \nu, m \mid \mathcal{N}(z)\, d\nu \tag{4.7}$$

and with the help of them we can achieve the coherent states quantization scheme described in Sec.2 to rederive the Berezin transform $\mathfrak{B}_m^{\nu,n}$ in (1.4) which was defined by Toeplitz operators

formalism in [13]. For this let us associate to any arbitrary function $\varphi \in L^2(\mathbb{B}^n, (1-|\xi|^2)^{-n-1}d\mu)$ the operator-valued integral

$$A_\varphi := \int_{\mathbb{B}^n} \mid z,\nu,m >< z,\nu,m \mid \varphi(z)\,\mathcal{N}(z)\,(1-|z|^2)^{-n-1}d\mu \tag{4.8}$$

The function $\varphi(z)$ is a upper symbol of the operator A_φ. On the other hand, we need to calculate the expectation value

$$\mathbb{E}_{\{|z,\nu,m>\}}\left(A_\varphi\right) := < z,\nu,m \mid A_\varphi \mid z,\nu,m > \tag{4.9}$$

of A_φ with respect to the set of coherent states $\{\mid z,\nu,m>\}_{z\in\mathbb{B}^n}$ defined in (4.1). This will constitute a lower symbol of the operator A_φ.

Proposition 4.2. *Let $\varphi \in L^2(\mathbb{B}^n, (1-|\xi|^2)^{-n-1}d\mu)$. Then, the expectation value in (4.9) has the following expression*

$$\begin{aligned}\mathbb{E}_{\{|z,\nu,m>\}}\left(A_\varphi\right) &= \frac{\Gamma(n)\,m!\,(2(\nu-m)-n)\,\Gamma(2\nu-m)}{n!\Gamma(n+m)\,\Gamma(2\nu-m-n+1)}\int_{\mathbb{B}}\left(\frac{(1-|z|^2(1-|\xi|^2))}{|1-\langle z,\xi\rangle|^2}\right)^{2(\nu-m)} \\ &\quad\times\left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2|\xi|^2\right)\right)^2\frac{\varphi(\xi)}{\left(1-|\xi|^2\right)^{n+1}}d\mu(\xi)\end{aligned} \tag{4.10}$$

for every $z \in \mathbb{B}^n$.

Proof. We first write the action of the operator A_φ in (4.8) on an arbitrary coherent state $\mid z,\nu,m>$ in terms of Dirac's bra-ket notation as

$$A_\varphi \mid z,\nu,m> = \int_{\mathbb{B}^n} \mid w,\nu,m>< w,\nu,m \mid z,\nu,m > \frac{\mathcal{N}(w)}{(1-|w|^2)^{n+1}}d\mu(w) \tag{4.11}$$

Therefore, the expectation value reads

$$< z,\nu,m \mid A_\varphi \mid z,\nu,m > = \int_{\mathbb{B}^n} < z,\nu,m \mid w,\nu,m > \overline{< z,\nu,m \mid w,\nu,m >}\frac{\mathcal{N}(w)}{(1-|w|^2)^{n+1}}d\mu(w) \tag{4.12}$$

$$= \int_{\mathbb{B}^n} |< z,\nu,m \mid w,\nu,m >|^2\,\varphi(w)\,\frac{\mathcal{N}(w)}{(1-|w|^2)^{n+1}}d\mu(w). \tag{4.13}$$

Now, we need to evaluate the quantity $|< z,\nu,m \mid w,\nu,m >|^2$ in (4.13). For this, we write the scalar product as

$$< z,\nu,m \mid w,\nu,m > = \sum_{p=0}^{+\infty}\sum_{q=0}^{m}\sum_{j=1}^{d(n;p,q)}\sum_{r=0}^{+\infty}\sum_{s=0}^{m}\sum_{l=1}^{d(n;p,q)}\frac{\Phi_{p,q,j}^{\nu,m}(z)\,\overline{\Phi_{p,q,j}^{\nu,m}(w)}}{\sqrt{\mathcal{N}(z)\,\mathcal{N}(w)}}\left\langle\varphi_{p,q,j},\varphi_{p,q,l}\right\rangle_{\mathcal{H}} \tag{4.14}$$

Recalling that

$$\left\langle\varphi_{p,q,j},\varphi_{p,q,l}\right\rangle_{\mathcal{H}} = \delta_{j,l}\delta_{p,r}\delta_{q,s} \tag{4.15}$$

since $\left\{\varphi_{p,q,j}\right\}$ is an orthonormal basis of $\mathcal{H}$, the above sum in (4.14) reduces to

$$< z,\nu,m \mid w,\nu,m > = (\mathcal{N}(z)\,\mathcal{N}(w))^{-\frac{1}{2}}\sum_{\substack{0\le q\le m, 0\le p<+\infty \\ 1\le j\le d(n,p,q)}}\Phi_{p,q,j}^{\nu,m}(z)\,\overline{\Phi_{p,q,j}^{\nu,m}(w)}. \tag{4.16}$$

Now, taking account (3.9) and (3.11), Equation (4.16) takes the form

$$< z,\nu,m \mid w,\nu,m > = \frac{(2(\nu-m)-n)\,\Gamma(2\nu-m)}{n!\Gamma(2\nu-m-n+1)}\left(\mathcal{N}(z)\,\mathcal{N}(w)\right)^{-\frac{1}{2}}\left(\frac{1-\overline{\langle z,w\rangle}}{1-\langle z,w\rangle}\right)^{\nu} \tag{4.17}$$
$$\times\left(\cosh\left(d\left(z,w\right)\right)\right)^{-2(\nu-m)} P_m^{(n-1,2(\nu-m)-n)}\left(1-2\tanh^2\left(d\left(z,w\right)\right)\right).$$

So that the square modulus of the scalar product in (4.17) reads

$$|< z,\nu,m \mid w,\nu,m >|^2 = \left(\frac{(2(\nu-m)-n)\,\Gamma(2\nu-m)}{n!\Gamma(2\nu-m-n+1)}\right)^2\left(\mathcal{N}(z)\,\mathcal{N}(w)\right)^{-1} \tag{4.18}$$
$$\times\left(\cosh\left(d\left(z,w\right)\right)\right)^{-4(\nu-m)}\left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2\tanh^2\left(d\left(z,w\right)\right)\right)\right)^2.$$

Returning back to (4.12) , we get

$$\mathbb{E}_{\{|z,\nu,m>\}}\left(A_\varphi\right) = \int_{\mathbb{B}^n}\varphi(w)\left(\frac{(2[\nu-m]-n)\,\Gamma(2\nu-m)}{n!\Gamma(2\nu-m-n+1)}\right)^2\left(\mathcal{N}(z)\,\mathcal{N}(w)\right)^{-1}\frac{\mathcal{N}(w)}{(1-|w|^2)^{n+1}} \tag{4.19}$$
$$\times\left(\cosh\left(d\left(z,w\right)\right)\right)^{-4(\nu-m)}\left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2\tanh^2\left(d\left(z,w\right)\right)\right)\right)^2 d\mu(w),$$

which can be also written as

$$\mathbb{E}_{\{|z,\nu,m>\}}\left(A_\varphi\right) = \int_{\mathbb{B}^n}\varphi(w)\left(\frac{(2(\nu-m)-n)\,\Gamma(2\nu-m)}{n!\Gamma(2\nu-m-n+1)}\right)^2\frac{(\mathcal{N}(z))^{-1}}{(1-|w|^2)^{n+1}} \tag{4.20}$$
$$\times\left(\cosh\left(d\left(z,w\right)\right)\right)^{-4(\nu-m)}\left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2\tanh^2\left(d\left(z,w\right)\right)\right)\right)^2 d\mu(w)$$
$$= \frac{(2(\nu-m)-n)\,\Gamma(2\nu-m)\,m!\Gamma(n)}{n!\Gamma(2\nu-m-n+1)\,\Gamma(m+n)}\int_{\mathbb{B}^n}\frac{\varphi(w)}{(1-|w|^2)^{n+1}} \tag{4.21}$$
$$\times\left(\cosh\left(d\left(z,w\right)\right)\right)^{-4(\nu-m)}\left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2\tanh^2\left(d\left(z,w\right)\right)\right)\right)^2 d\mu(w).$$

Finally, we summarize the above discussion by considering the following definition.

Definition 4.3. *The Berezin transform of the classical observable $\varphi \in L^2(\mathbb{B}^n,(1-|\xi|^2)^{-n-1}d\mu)$ constructed according to the quantization by the coherent states $\{\mid z,\nu,m>\}$ in (4.1) is obtained by associating to φ the obtained mean value in (4.10) . That is,*

$$\mathfrak{B}_m^{\nu,n}[\varphi](z) = \mathbb{E}_{\{|z,\nu,m>\}}\left(A_\varphi\right) \tag{4.22}$$

for every $z \in \mathbb{B}^n$.

Remark 4.4. For $m=0$, the transform (4.10) reduces to the well known Berezin transform attached to the weighted Bergman space $\mathcal{A}_0^{2,\nu}(\mathbb{B}^n)$ of holomorphic function ψ on $\mathbb{B}^n$ satisfying the growth condition (3.12) and given by

$$\mathfrak{B}_0^{\nu,n}[\varphi](z) = \frac{(2\nu-n)\,\Gamma(2\nu)}{n!\Gamma(2\nu-n+1)}\int_{\mathbb{B}^n}\left(\cosh d(z,\xi)\right)^{-4\nu}\frac{\varphi(\xi)}{\left(1-|\xi|^2\right)^{n+1}}d\mu(\xi) \tag{4.23}$$

The latter one have also been written as a function of the Bergman Laplacian $\Delta_{\mathbb{B}^n}$ as

$$\mathfrak{B}_0^{\nu,n} = \frac{1}{\Gamma(\alpha+1)\,\Gamma(\alpha+n+1)}\left|\Gamma\left(\alpha+1+\frac{n}{2}+\frac{i}{2}\sqrt{-\Delta_{\mathbb{B}^n}-n^2}\right)\right|^2 \tag{4.24}$$

firstly by Berezin. The above form, involving gamma factors, was derived by Peetre in [18, p. 182], so that α there occurring in the weight of the Bergman space, corresponds to $2\nu-n-1$.

5 An expression of $\mathfrak{B}_m^{\nu,n}$ as function of $\Delta_{\mathbb{B}^n}$

Then Berezin transform $\mathfrak{B}_m^{\nu,n}$ associated the generalized Bergman space $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$ is given by

$$\mathfrak{B}_m^{\nu,n}[\varphi](z) = c_m^{\nu,n} \int_{\mathbb{B}^n} \frac{\left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2\tanh^2 d(z,\xi)\right)\right)^2}{(\cosh d(z,\xi))^{4(\nu-m)}} \varphi(\xi) \frac{d\mu(\xi)}{\left(1-|\xi|^2\right)^{n+1}}, \tag{5.1}$$

with

$$c_m^{\nu,n} = \frac{\Gamma(n)\, m!\, (2(\nu-m)-n)\,\Gamma(2\nu-m)}{n!\Gamma(n+m)\,\Gamma(2\nu-m-n+1)} \tag{5.2}$$

Let $B_m^{\nu,n}(z,w)$ be the kernel function of the above integral operator and set $h_m^{\nu,n}(g) = B_m^{\nu,n}(z,0)$, $z = g.0$. Then the integral operator (5.1) can be written as a convolution product over G:

$$\mathfrak{B}_m^{\nu,n}[\varphi](z) = c_m^{\nu,n}(\varphi * h_m^{\nu,n})(g), \quad z = g.0,$$

from which it follows easily that the Berezin operator is an L^2-bounded operator.
Since $B_m^{\nu,n}(z,w)$ is a G bi-invariant function it follows that $\mathfrak{B}_m^{\nu,n}$ is a G-invariant operator. That is $U(g)\circ\mathfrak{B}_m^{\nu,n} = \mathfrak{B}_m^{\nu,n}\circ U(g)$, for every $g \in G$. Therefore $\mathfrak{B}_m^{\nu,n}$ is, in the spectral theoretic sense, a function of the G-invariant Laplacian $\Delta_{\mathbb{B}^n}$ of the unit ball. Below we give it explicitly.

Theorem 5.1. *The Berezin transform $\mathfrak{B}_m^{\nu,n}$ can be expressed as a function of the Laplace-Beltrami operator $\Delta_{\mathbb{B}^n}$ as*

$$\begin{aligned}\mathfrak{B}_m^{\nu,n} = &\left|\Gamma\left(2(\nu-m)-\frac{n}{2}+\frac{i}{2}\sqrt{-\Delta_{\mathbb{B}^n}-n^2}\right)\right|^2 \\ &\sum_{k=0}^{2m}\gamma_k^{\nu,m,n} W_k(-\frac{1}{4}\Delta_{\mathbb{B}^n}-\frac{n^2}{4};2(\nu-m)-\frac{n}{2},1+\frac{n}{2},\frac{n}{2},\frac{n}{2})\end{aligned} \tag{5.3}$$

where $W_k(.)$ are Wilson polynomials,

$$\gamma_k^{\nu,n,m} = \frac{2m!\Gamma(n)\,(2(\nu-m)-n)\,\Gamma(2\nu-m)\,(-1)^k}{\Gamma(n+m)\,\Gamma(2\nu-m-n+1)\,k!\Gamma^2(2(\nu-m)+k)} \times A_k^{\nu,n,m},$$

and the coefficients $A_k^{\nu,n,m}$ are given by (5.10) below.

Proof. Recall that

$$\mathfrak{B}_m^{\nu,n}[\varphi](z) = c_m^{\nu,n}(\varphi * h_m^{\nu,n})(g), \quad z = g.0,$$

where

$$h_m^{\nu,n}(\xi) := \left(1-|\xi|^2\right)^{2(\nu-m)}\left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2|\xi|^2\right)\right)^2, \xi\in\mathbb{B}^n, \tag{5.4}$$

By this way, we have to compute the spherical transform $\mathcal{F}[h_m^{\nu,n}]$ of $h_m^{\nu,n}$. Namely

$$\mathcal{F}[h_m^{\nu,n}](\lambda) := \int_{\mathbb{B}^n} h_m^{\nu,n}(z)\phi_{-\lambda}(z)d\mu_n(z), \lambda\in\mathbb{R} \tag{5.5}$$

where ϕ_λ is the spherical function associated to $\Delta_{\mathbb{B}^n}$, given by

$$\phi_\lambda(z) = (1-|z|)^{\frac{i\lambda+n}{2}} \quad {}_2F_1(\frac{i\lambda+n}{2},\frac{i\lambda+n}{2},n;|z|^2).$$

Using Pfaff's transformation [14]

$${}_2F_1(a,b,c;x) = (1-x)^{-b}\ {}_2F_1\left(b,c-a,c;\frac{x}{x-1}\right) \tag{5.6}$$

we rewrite $\phi_{-\lambda}$ as

$$\phi_{-\lambda}(z) = \quad {}_2F_1\left(\frac{-i\lambda+n}{2},\frac{i\lambda+n}{2},n;\frac{|z|^2}{|z|^2-1}\right) \tag{5.7}$$

So that returning back to (5.5) we get

$$\mathcal{F}[h_m^{\nu,n}](\lambda) = 2n \int_0^1 \frac{\rho^{2n-1}}{(1-\rho^2)^{n+1-2(\nu-m)}} \left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2\rho^2\right)\right)^2 \tag{5.8}$$

$$\times {}_2F_1\left(\frac{n+i\lambda}{2}, \frac{n-i\lambda}{2}, n; \frac{\rho^2}{\rho^2-1}\right) d\rho$$

To calculate this last integral, we first use a linearisation of the square of Jacobi polynomial in (5.8) by making appeal to the following Clebsh-Gordon type formula see [8, p. 611],

$$P_s^{(\kappa,\epsilon)}(u)\, P_l^{(\tau,\eta)}(u) = \sum_{k=0}^{s+l} A_{s,l}(k)\, P_k^{(\alpha,\delta)}(u) \tag{5.9}$$

for the particular case of parameters $s = l = m$, $\kappa = \tau = \alpha = n-1$, $\epsilon = \eta = 2(\nu-m)-n$ and $\delta = 2(\nu-m)$. In our setting, the linearisation coefficients $A_{s,l}(k)$ are of the form

$$A_k^{\nu,n,m} = \frac{(2(\nu-m)+n)_k\,(n)_{2m}\,(2k+2(\nu-m)+n)\,(-1)^k\,(2m)!\,((2(\nu-m))_{2m})^2}{(n)_k\,(2(\nu-m)+n)_{2m+k+1}\,(m!)^2\,(2m-k)!\,((2(\nu-m))_m)^2} \tag{5.10}$$

$$\times F_{2:1}^{2:2}\left(\begin{array}{c} -2m+k, -2\nu-k-n : -m, -n-m+1; -m, -m-n+1 \\ -2m, -2m-n+1 : 1-2\nu, 1-2\nu \end{array} \mid 1,1\right)$$

Here $F_{l:l'}^{p:p'}(.)$ denotes the Kampé de Fériet double hypergeometric function defined by [20, p. 63]

$$F_{l:l'}^{p:p'}\left(\begin{array}{c} (a_p) : (b_{p'}), (c_{p'}) \\ (d_l) : (\kappa_{l'}), (\varrho_{l'}) \end{array} \mid x, y\right) = \sum_{q,s=0}^{+\infty} \frac{[a_p]_{q+s}\,[b_{p'}]_q\,[c_{p'}]_s}{[d_l]_{q+s}\,[\kappa_{l'}]_q\,[\varrho_{l'}]_s} \frac{x^q}{q!}\frac{y^s}{s!} \tag{5.11}$$

where $[a_p]_s = \prod_{j=1}^p (a_j)_s$ in which $(x)_s = x(x+1)\ldots(x+s-1)$ is the Pochhammer symbol. Therefore, inserting

$$\left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2\rho^2\right)\right)^2 = \sum_{k=0}^{2m} A_k^{\nu,n,m} P_k^{(n-1,2(\nu-m))}\left(1-2\rho^2\right) \tag{5.12}$$

into (5.8) the Fourier-Helgason transform of $h_m^{\nu,n}$ takes the form

$$\mathcal{F}[h_m^{\nu,n}](\lambda) = \sum_{k=0}^{2m} A_k^{\nu,n,m} \mathfrak{I}_k^{\nu,n,m}(\lambda), \tag{5.13}$$

where the last term in this sum is defined by the integral

$$\mathfrak{I}_k^{\nu,m}(\lambda) = \int_0^1 \frac{2n\rho^{2n-1}}{(1-\rho^2)^{n+1-2(\nu-m)}} P_k^{(n-1,2(\nu-m))}\left(1-2\rho^2\right) \tag{5.14}$$

$$\times {}_2F_1\left(\frac{1}{2}(n+i\lambda), \frac{1}{2}(n-i\lambda), n; \frac{\rho^2}{\rho^2-1}\right) d\rho$$

To calculate this last integral we make the change of variable $\rho = \tanh t$. Therefore (5.14) reads

$$\mathfrak{I}_k^{\nu,m}(\lambda) = \int_0^{+\infty} 2n\,(\sinh t)^{2n-1}\, P_k^{(n-1,2(\nu-m))}\left(1-2\tanh^2 t\right) \tag{5.15}$$

$$\times (\cosh t)^{-4(\nu-m)+1} . {}_2F_1\left(\frac{n+i\lambda}{2}, \frac{n-i\lambda}{2}, n; -\sinh^2 t\right) dt$$

Now, we make use of the result established by Koornwinder [17]

$$\int_{0}^{+\infty}(\cosh t)^{-\alpha+\beta-\delta-\mu'-1}(\sinh t)^{2\alpha+1}P_k^{(\alpha,\delta)}\left(1-2\tanh^2 t\right)$$
$$\times{}_2F_1\left(\frac{\alpha+\beta+1+i\lambda}{2},\frac{\alpha+\beta+1-i\lambda}{2},\alpha+1;-\sinh^2 t\right)dt$$
$$=\frac{\Gamma(\alpha+1)(-1)^k\Gamma\left(\frac{1}{2}(\delta+\mu'+1+i\lambda)\right)\Gamma\left(\frac{1}{2}(\delta+\mu'+1-i\lambda)\right)}{k!\Gamma\left(\frac{1}{2}(\alpha+\beta+\delta+\mu'+2)+k\right)\Gamma\left(\frac{1}{2}(\alpha-\beta+\delta+\mu'+2)+k\right)} \tag{5.16}$$
$$\times W_k\left(\frac{1}{4}\lambda^2;\frac{1}{2}(\delta+\mu'+1),\frac{1}{2}(\delta-\mu'+1),\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha-\beta+1)\right)$$

where $\beta,\delta,\lambda\in\mathbb{R}$, $\alpha,\delta>-1$, $\delta+\Re(\mu)'>-1$ and $W_k(.)$ is the Wilson polynomial given in terms of the ${}_4F_3$-sum as ([1],p. 158):

$$W_k\left(x^2,a,b,c,d\right):=(a+b)_k(a+c)_k(a+d)_k$$
$$\times{}_4F_3\left(\begin{matrix}-k,k+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{matrix}\mid 1\right) \tag{5.17}$$

for the parameters $\alpha=n-1,\delta=2(\nu-m)-n,\beta=0$ and $\mu'=2(\nu-m)-n-1$.We find that

$$\mathfrak{I}_k^{\nu,m}(\lambda)=\frac{2n\Gamma(n)(-1)^k}{k!\Gamma^2(2(\nu-m)+k)}\left|\Gamma\left(2(\nu-m)-\frac{n}{2}+i\frac{\lambda}{2}\right)\right|^2 \tag{5.18}$$
$$\times W_k\left(\frac{1}{4}\lambda^2;2(\nu-m)-\frac{n}{2},1+\frac{n}{2},\frac{n}{2},\frac{n}{2}\right)$$

Summarizing the above calculations

$$\mathcal{F}\left[h_m^{\nu,n}\right](\lambda)=\left|\Gamma\left(2(\nu-m)-\frac{n}{2}+i\frac{\lambda}{2}\right)\right|^2 \tag{5.19}$$
$$\times\sum_{k=0}^{2m}\gamma_k^{\nu,n,m}W_k\left(\frac{\lambda^2}{4};2(\nu-m)-\frac{n}{2},1+\frac{n}{2},\frac{n}{2},\frac{n}{2}\right)$$

with the constants

$$\gamma_k^{\nu,n,m}:=\frac{2m!\Gamma(n)(2(\nu-m)-n)\Gamma(2\nu-m)(-1)^k A_k^{\nu,n,m}}{\Gamma(n+m)\Gamma(2\nu-m-n+1)k!\Gamma^2(2(\nu-m)+k)}, \tag{5.20}$$

where the constants $A_k^{\nu,n,m}$ is given by (5.10). Finally, replacing λ by $\sqrt{-\Delta_{\mathbb{B}^n}-n^2}$, we arrive at the announced result.

Remark 5.1. Setting $m=0$ in the formula (5.3) in Theorem 5.1, we recover the result of Peetre [18].

Remark 5.2. We should note that the transform $\mathfrak{B}_m^{\nu,n}$ have been expressed in [13] as a function of the Laplace-Beltrami operator $\Delta_{\mathbb{B}^n}$ in terms of the ${}_3F_2$-sum as

$$\mathfrak{B}_m^{\nu,n}=\sum_{j=0}^{2m}C_j^{\nu,n,m}\frac{\Gamma\left(2(\nu-m)-\frac{1}{2}\left(n-i\sqrt{-\Delta_{\mathbb{B}^n}-n^2}\right)\right)}{\Gamma\left(2(\nu-m)+j+\frac{1}{2}\left(n+i\sqrt{-\Delta_{\mathbb{B}^n}-n^2}\right)\right)} \tag{5.21}$$
$$\times{}_3F_2\left[\begin{matrix}\frac{1}{2}\left(n+i\sqrt{-\Delta_{\mathbb{B}^n}-n^2}\right),n+j,\frac{1}{2}\left(n+i\sqrt{-\Delta_{\mathbb{B}^n}-n^2}\right)\\ (\nu-m)+j+\frac{1}{2}\left(n+i\sqrt{-\Delta_{\mathbb{B}^n}-n^2}\right),n\end{matrix}\mid 1\right]$$

where

$$C_j^{\nu,n,m} = \frac{(2(\nu-m)-n)\,\Gamma(n+m)\,(-1)^j\,\Gamma(n+j)}{m!\Gamma(2\nu-n-m+1)\,\Gamma(2\nu-n)} \tag{5.22}$$
$$\times \sum_{p=\max(0,j-m)}^{\min(m,j)} \frac{(m!)^2\,\Gamma(2\nu-m)\,\Gamma(2\nu-m+j-p)}{(j-p)!\,(m+p-j)!p!\,(m-p)!\Gamma(n+j-p)\,\Gamma(n+p)}.$$

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special Functions,* Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1999

[2] Ayaz, K.and Intissar, A.: Selberg trace formula for the heat and wave kernels of Maass Lpalacians of the complex hyperbolic space $H^n(\mathbb{C})$, $n \geq 2$. *Diff. Geom. App.* **15**, 1-31 (2001)

[3] Berezin, F.A.: General concept of quantization. Commun. Math. Phys. **40**, 153-174 (1975)

[4] Berezin, F.A.: Quantization. Math. USSR, Izv. **8**, 1109-1165 (1974)

[5] Berezin, F.A.: Quantization in complex symmetric spaces. Izv. Akad. Naud.SSSR. Ser.Mat(2) **39**, 1109-1165 (1974)

[6] Bonami, N. Bruna, J. and Grellier, S.: On Hardy, BMO and Lipschitz spaces of invariantly harmonic functions in the unit ball. *Proc. London Math. Soc.* **3**, 665-696 (1998)

[7] Boussejra, A. and Intissar, A. L^2-concrete Spectral Analysis of the Invariant Laplacian $\Delta_{\alpha,\beta}$ in the Unit complex ball $\mathbb{B}^n$, J. Funct. Anal 160, 115-140 (1998)

[8] Chaggara, H. and Koepf, W.: On linearization coefficients of Jacobi polynomials. *App. Math. Lett.* **23**, 609-614 (2010)

[9] Dodonov, V. V. : 'Noncalssical' states in quantum optics: a 'squeezed review of the first 75 years, *J.opt.B:Quantum Semiclass.opt.***4**, R1-R33 (2002)

[10] Folland, G. B.: Spherical harmonic expansion of the Poisson-Szegö kernel for the ball. *Proc.Am. Math. Soc.* **47** (2) , 401-407 (1975)

[11] Gazeau, J.P.: Coherent states in quantum physics. Wiley, Weinheim (2009)

[12] Geller, D.: Some results in H^p theory for the Heisenberg group, *Duke Math. J.* **47**, 365-391 (1980)

[13] Ghanmi, A. and Mouayn, Z.: A formula representing magnetic Berezin transforms on the Bergman ball of $\mathbb{C}^n$ as functions of the Laplace-Beltrami operator, *arxiv: 1101.1248v1* (2011)

[14] Gradshteyn I S and Ryzhik I M, "Table of Integrals, Series and Products", Academic Press, INC, Seven Edition 2007

[15] Helgason, S.: A duality for symmetric spaces with applications to group representations. *Advances in math.*, **5**, 1-154 (1970)

[16] Mourad E.H.Ismail.: Classical and Quantum Orthogonal Polynomials in one variable, Encyclopedia of Mathematics and its applications, Cambridge university press (2005)

[17] Koornwinder T.H.: Special orthogonal polynomial systems mapped onto each other by the Fourier-Jacobi transform. *Orthogonal polynomials and applications* (Bar-le-Duc, 1984), 174-183, Lecture Notes in Math. **1171**, Springer, Berlin, 1985

[18] Peetre, J.: The Berezin transform and Ha-plitz operator. J. Operator Theory. **24**, 165-186 (1990)

[19] Rudin, W.: Function theory in the unit ball of $\mathbb{C}^n$. Grundlehern der mathematischen Wissenschften, vol.241. Springer, New York (1980)

[20] Srivastava, H. M. and Manocha A.: A treatise on Generating functions, John Willey and Sons, New York, Chichester, Brisbane, Toraonto (1984)

[21] Unterberger, A., Upmeir, H.: The Berezin transform and invariant differential operators. *Commun. Math. Phys.* **164**, 563-597 (1994)

[22] Zernik, F. and Brinkman, H.C.: Hyperspharische Functionen und die in spahrischen bereichen orthogonalen polynome. Proc. Kon. Akad. v. wet. Amsterdam 38, 161-170 (1935)

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